THREE-BODY MATRIX ELEMENTS FOR CALCULATIONS OF MEAN FIELD AND exp(S) GROUND STATE CORRELATIONS.*

by

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Abstract. In this document we present our approach to the computation of three-body matrix elements, based on the Urbana family of three-nucleon potentials. The calculations refer only to the necessary matrix elements needed to include the three-nucleon interaction in the manner presented in reference [1].

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1. Three-Nucleon Interaction $\mathbf{V}^{tni}$.

In the description of the Urbana potential series [2], the three-nucleon interaction is presented as a sum of a long-range part derivable from the two-pion exchange diagrams, and a phenomenological short-range part, adjusted to reproduce the binding energy of the three-body nuclear system. The two-pion exchange interaction ($V_{2\pi,tni}$) is given as

$$\begin{aligned} V_{2\pi,tni} = \sum_{cycl.} & A_{2\pi}\{\tau_1\cdot,\tau_2,\tau_1\cdot\tau_3\}\{S_{12}T(r_{12})+\sigma_1\cdot\sigma_2 Y(r_{12})),S_{13}T(r_{13})+\sigma_1\cdot\sigma_3 Y(r_{13}))\} \\ & C_{2\pi}[\tau_1\cdot,\tau_2,\tau_1\cdot\tau_3][S_{12}T(r_{12})+\sigma_1\cdot\sigma_2 Y(r_{12})),S_{13}T(r_{13})+\sigma_1\cdot\sigma_3 Y(r_{13}))] \end{aligned} \tag{1.1}$$

Here $\sum_{cycl.}$ represents a sum over cyclic permutations over the indices 1,2, and 3. τ, σ, and S_{ij} are the isospin, spin, and tensor operators, and $\{,\}$ and $[,]$ denote the anticommutators and commutators. The $T(r)$ and $Y(r)$ are radial functions associated with the tensor and Yukawa parts of the one-pion-exchange interaction, and $C_{2\pi}=\frac{1}{4}A_{2\pi}$.

The short range repulsion is phenomenological and is given as

$$V_{R,tni} = U_0\sum_{cycl.} T^2(r_{12})T^2(r_{13}) \tag{1.2}$$

Note. For the Urbana IX potential, the fitted parameters are: $A_{2\pi}=-0.0293$ and $U_0=+0.0048$.

The relevant matrix elements for our calculation are of two types only. The matrix elements of the form $V^{tni,a}_{h,a_1,a_2;h,b_1,b_2}$ are derived in section 2, whereas the matrix elements of the form $-V^{tni,a}_{h,a_1,a_2;b_1,h,b_2}$ are presented in section 3.

2. Density-Dependent Matrix Elements.

In this section we work out the computation of the integrals

$$V^{tni}_{h,a_1,b_1,p,a_2,b_2} =< \phi_h(1)\phi_{a_1}(2)\phi_{b_1}(3)|V^{tni}|\phi_p(1)\phi_{a_2}(2)\phi_{b_2}(3) > . \tag{2.1}$$

We employ the same methods as developed for the calculations of the two-body matrix elements, namely using Fourier-transforms in order to separate the variables. We use an expansion into Harmonic oscillator functions $H^{\ell}_n(r)$ for which the Fourier transforms are again Harmonic oscillator functions $H^{\ell}_n(q)$.

We will make use of the particular angular momentum coupling in order to make the computations feasible. For the matrix elements given by (2.1) and requiring $j_h=j_p$ the angular momentum coupling is similar to that of our two-body matrix elements:

$$\begin{aligned} < (a_1\bar a_2)_\lambda|V^{eff}|(b_2\bar b_1)_\lambda > =& \delta_{m_h,m_p}\ (-)^{k_{a_1}+k_{b_2}}(-)^{(k_{a_2}+k_{b_1}+m_{a_2}-m_{b_1})} \\ & < j_{a_1}m_{a_1}j_{a_2}-m_{a_2}|\lambda\mu >< j_{b_2}m_{b_2}j_{b_1}-m_{b_1}|\lambda\mu > V_{m_h m_{a_1} m_{b_1},m_p m_{a_2} m_{b_2}} \\ < (a_1\bar a_2)_\lambda|V^{eff,x}|(b_2\bar b_1)_\lambda > =& \delta_{m_h,m_p}\ (-)^{k_{a_1}+k_{b_2}}(-)^{(k_{a_2}+k_{b_1}+m_{a_2}-m_{b_1})} \\ & < j_{a_1}m_{a_1}j_{a_2}-m_{a_2}|\lambda\mu >< j_{b_2}m_{b_2}j_{b_1}-m_{b_1}|\lambda\mu > (-)V_{m_h m_{a_1} m_{b_1},m_{a_2} m_p m_{b_2}} \end{aligned} \tag{2.2}$$

which includes the "Ring"-phase for two-body ph-matrix elements. Here we do sum over $m_h=m_p$ and we do sum over all other m's. This angular momentum coupling applies as well for the matrix elements of section 3. In this section we handle the sum over the cyclic permutations by calculating the three matrix elements separately

$$\begin{aligned} V^{tni}_{h,a_1,b_1,p,a_2,b_2} = & < \phi_h(1)\phi_{a_1}(2)\phi_{b_1}(3)|V^{tni}|\phi_p(1)\phi_{a_2}(2)\phi_{b_2}(3) > \\ & < \phi_h(2)\phi_{a_1}(3)\phi_{b_1}(1)|V^{tni}|\phi_p(2)\phi_{a_2}(3)\phi_{b_2}(1) > \\ & < \phi_h(3)\phi_{a_1}(1)\phi_{b_1}(2)|V^{tni}|\phi_p(3)\phi_{a_2}(1)\phi_{b_2}(2) > \end{aligned} \tag{2.3}$$

2a. Short range repulsion term.

As this term does not contain any additional operators it is particularly simple to handle. We use Eq. (2.2) of Reference [3]

$$T^2(r_{12}) = 4\pi \ \frac{2}{\pi} \int q^2 dq \tilde{T}(q) \sum_{\ell} (-)^{\ell} \hat{\ell} \ j_{\ell}(qr_1) j_{\ell}(qr_2) \big[Y_{\ell}(\hat{r}_1) \otimes Y_{\ell}(\hat{r}_2) \big]^{(0)} \tag{2.4}$$

using

$$\tilde{T}(q) = \int r_{12}^2 dr_{12} T^2(r_{12}) j_0(qr_{12}) \tag{2.5}$$

Correspondingly we write

$$\begin{aligned} T^2(r_{12})T^2(r_{13}) =& (4\pi)^2 \sum_{\ell_2,\ell_3} (-)^{(\ell_2+\ell_3)} \hat{\ell}_2 \hat{\ell}_3 \\ & \times \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_{\ell_2}(q_2 r_1) j_{\ell_2}(q_2 r_2) \times \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell_3}(q_3 r_1) j_{\ell_3}(q_3 r_3) \\ & \times \big[Y_{\ell_2}(\hat{r}_1) \otimes Y_{\ell_2}(\hat{r}_2) \big]^{(0)} \big[Y_{\ell_3}(\hat{r}_1) \otimes Y_{\ell_3}(\hat{r}_3) \big]^{(0)} \end{aligned} \tag{2.6}$$

recoupling the spherical harmonics results in

$$\begin{aligned} T^2(r_{12})T^2(r_{13}) =& (4\pi)^2 \frac{1}{\sqrt{4\pi}} \sum_{\ell_1,\ell_2,\ell 3} (-)^{(\ell_2+\ell_3)} \hat{\ell}_2 \hat{\ell}_3 \\ & \times \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_{\ell_2}(q_2 r_1) j_{\ell_2}(q_2 r_2) \times \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell_3}(q_3 r_1) j_{\ell_3}(q_3 r_3) \\ & \times \Big[Y_{\ell_1}(\hat{r}_1) \otimes \big[Y_{\ell_2}(\hat{r}_2) \otimes Y_{\ell_3}(\hat{r}_3) \big]^{(\ell_1)} \Big]^{(0)} \end{aligned} \tag{2.7}$$

This leads to $\ell_1 = 0$ for the first matrix element, $\ell_2 = 0$ for the second and $\ell_3 = 0$ for the third. In turn we can write the interaction for the three matrix elements respectively as

$$\begin{aligned} T^2(r_{12})T^2(r_{13}) =& \sum_{\ell} (-)^{\ell} \hat{\ell} \ \big[\sqrt{4\pi} Y_{\ell}(\hat{r}_2) \odot \sqrt{4\pi} Y_{\ell}(\hat{r}_3) \big] \\ & \times \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_{\ell}(q_2 r_1) j_{\ell}(q_2 r_2) \times \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell}(q_3 r_1) j_{\ell}(q_3 r_3) \end{aligned} \tag{2.8}$$

for the first matrix element and

$$\begin{aligned} T^2(r_{12})T^2(r_{13}) =& \sum_{\ell} \ \big[\sqrt{4\pi} Y_{\ell}(\hat{r}_1) \odot \sqrt{4\pi} Y_{\ell}(\hat{r}_3) \big] \\ & \times \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_0(q_2 r_1) j_0(q_2 r_2) \times \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell}(q_3 r_1) j_{\ell}(q_3 r_3) \\ T^2(r_{12})T^2(r_{13}) =& \sum_{\ell} \ \big[\sqrt{4\pi} Y_{\ell}(\hat{r}_1) \odot \sqrt{4\pi} Y_{\ell}(\hat{r}_2) \big] \\ & \times \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_0(q_3 r_1) j_0(q_3 r_3) \times \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_{\ell}(q_2 r_1) j_{\ell}(q_2 r_2) \end{aligned} \tag{2.9}$$

for the second and third respectively.

For the first matrix element we write

$$\begin{aligned} < \phi_h(1)\phi_{a_1}(2)\phi_{b_1}(3) | T^2(r_{12})T^2(r_{13}) | \phi_p(1)\phi_{a_2}(2)\phi_{b_2}(3) >=& (-)^{\lambda} \hat{\lambda} \int r_1^2 dr_1 R_h(r_1) R_p(r_1) \\ & \times (-)^{k_{a_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{a_1} \| Y_{\lambda} \| j_{a_2} > \int r_2^2 dr_2 R_{a_1}(r_2) R_{a_2}(r_2) \frac{2}{\pi} \int q_2^2 dq_2 \tilde{T}(q_2) j_{\lambda}(q_2 r_1) j_{\lambda}(q_2 r_2) \\ & \times (-)^{k_{b_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{b_2} \| Y_{\lambda} \| j_{b_1} > \int r_3^2 dr_3 R_{b_1}(r_3) R_{b_2}(r_3) \frac{2}{\pi} \int q_3^2 dq_3 \tilde{T}(q_3) j_{\lambda}(q_3 r_1) j_{\lambda}(q_3 r_3) \end{aligned} \tag{2.10}$$

We now assume $R_{a_1}(r)R_{a_2}(r)$ is expanded into harmonic oscillator functions as

$$R_{a_1}(r_2)R_{a_2}(r_2) = \sum_n A_n^a H_n^\lambda(r_2) \tag{2.11}$$

Using Gauss integration the coefficients can be written in terms of the functions at the Gauss-points, r_i, with weights, w_i, as

$$A_n^a = \sum_i \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{H}_n^\lambda(r_i)r_i^2 w_i \tag{2.12}$$

where the barred functions are those without the exponentials. In turn, we can write the integral

$$\begin{aligned} \int r_2^2 dr_2 R_{a_1}(r_2)R_{a_2}(r_2)\frac{2}{\pi}\int q_2^2 dq_2 \tilde{T}(q_2) j_\lambda(q_2 r_1) j_\lambda(q_2 r_2) = \\ \sum_{n,m} A_n^a T_{m,n}^\lambda H_m^\lambda(r_1) \end{aligned} \tag{2.13}$$

where

$$T_{m,n}^\lambda = \int q_2^2 dq_2 H_m^\lambda(q_2)\tilde{T}(q_2)H_n^\lambda(q_2) = \sum_s q_s^2 w_s \bar{H}_m^\lambda(q_s)\tilde{T}(q_s)\bar{H}_n^\lambda(q_s) \tag{2.14}$$

We proceed similarly with the integration over q_3 and r_3. Thus this matrix element can be written as

$$\begin{aligned} &< \phi_h(1)\phi_{a_1}(2)\phi_{b_1}(3)|T^2(r_{12})T^2(r_{13})|\phi_p(1)\phi_{a_2}(2)\phi_{b_2}(3) >= \\ &\quad (-)^{k_{b_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{a_1}\|Y_\lambda\|j_{a_2} > (-)^{k_{b_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{b_2}\|Y_\lambda\|j_{b_1} > \\ &\quad \times \sum_{i,j}\bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)\Delta S_{i,j} \end{aligned} \tag{2.15}$$

where

$$\Delta S_{i,j} = \bar{H}_n^\lambda(r_i)r_i^2 w_i \bar{H}_k^\lambda(r_j)r_j^2 w_j T_{n,m}^\lambda T_{k,l}^\lambda D_{m,l}^\lambda \tag{2.16}$$

with

$$D_{m,l}^\lambda = (-)^\lambda \hat{\lambda}\int r_s^2 dr_s H_m^\lambda(r_s)H_l^\lambda(r_s)R_p(r_s)R_h(r_s) = (-)^\lambda\hat{\lambda}\sum_s r_s^2 w_s \bar{H}_m^\lambda(r_s)\bar{H}_l^\lambda(r_s)R_p(r_s)R_h(r_s) \tag{2.17}$$

For the second matrix element we obtain

$$\begin{aligned} < \phi_{b_1}(1)\phi_h(2)\phi_{a_1}(3)|T^2(r_{12})T^2(r_{13})|\phi_{b_2}(1)\phi_p(2)\phi_{a_2}(3) >= \int r_2^2 dr_2 R_h(r_2)R_p(r_2) \\ \times(-)^{k_{a_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{b_1}\|Y_\lambda\|j_{b_2} > \int r_1^2 dr_1 R_{b_1}(r_1)R_{b_2}(r_1)\frac{2}{\pi}\int q_2^2 dq_2\tilde{T}(q_2)j_0(q_2 r_1)j_0(q_2 r_2) \\ \times(-)^{k_{a_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{a_2}\|Y_\lambda\|j_{a_1} > \int r_3^2 dr_3 R_{a_1}(r_3)R_{a_2}(r_3)\frac{2}{\pi}\int q_3^2 dq_3\tilde{T}(q_3)j_\lambda(q_3 r_1)j_\lambda(q_3 r_3) \end{aligned} \tag{2.18}$$

Again, this equivalent two-body matrix element takes the form

$$\begin{aligned} &< \phi_{b_1}(1)\phi_h(2)\phi_{a_1}(3)|T^2(r_{12})T^2(r_{13})|\phi_{b_2}(1)\phi_p(2)\phi_{a_2}(3) >= \\ &\quad (-)^{k_{b_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{b_1}\|Y_\lambda\|j_{b_2} > (-)^{k_{a_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}} < j_{a_2}\|Y_\lambda\|j_{a_1} > \\ &\quad \times\sum_{i,j}\bar{R}_{b_1}(r_i)\bar{R}_{b_2}(r_i)\bar{R}_{a_1}(r_j)\bar{R}_{a_2}(r_j)\Delta S_{i,j} \end{aligned} \tag{2.19}$$

In this case

$$\Delta S_{i,j} = r_i^2 w_i r_j^2 w_j H_m^0(r_i) T_{m,n}^0 C_n^0 \bar{H}_l^\lambda(r_j) \bar{H}_k^\lambda(r_i) T_{k,l}^\lambda \tag{2.20}$$

where

$$C_n^0 = \int r_2^2 dr_2 R_h(r_2) R_p(r_2) H_n^0(r_2) = \sum_k r_k^2 w_k \bar{R}_h(r_k) \bar{R}_p(r_k) \bar{H}_n^0(r_k) \tag{2.21}$$

Similarly, the third density term is

$$\begin{aligned} T^2(r_{12})T^2(r_{13}) =& \Big(\frac{1}{\pi}\Big)^3 \sum_\ell \left[Y_\ell(\hat{r}_1) \odot Y_\ell(\hat{r}_2)\right] \\ & \times \int q_2^2 dq_2 \tilde{T}(q_2) j_\ell(q_2 r_1) j_\ell(q_2 r_2) \times \int q_3^2 dq_3 \tilde{T}(q_3) j_0(q_3 r_1) j_0(q_3 r_3) \end{aligned} \tag{2.22}$$

The form can be obtained from the previous case by using the transpose i.e. $\Delta S^c_{i,j} = \Delta S^b_{j,i}$.

2b. Two-pion exchange term.

For the model V 2π-exchange TNI takes the form

$$V_{2\pi}^{tni,a} = A_{2\pi}\{\tau_1\tau_2, \tau_1\tau_3\}\{(S_{12}T(r_{12}) + \sigma_1\sigma_2 Y(r_{12})), (S_{13}T(r_{13}) + \sigma_1\sigma_3 Y(r_{13}))\} \tag{2.23}$$

We can write the tensor interaction as

$$S_{12} = 3 < 1010|K0 > \sqrt{4\pi}\Big[Y^{(K)}(\hat{r}_{12}) \otimes \left[\sigma_1 \otimes \sigma_2\right]^{(K)}\Big]^{(0)} \tag{2.24}$$

with $K = 2$, whereas the sigma interaction is identical with $K = 0$. By defining the form factors

$$4\pi\ V^K(q) = 4\pi \int V^K(r) j_K(qr) r^2 dr \tag{2.25}$$

with $V(r) = Y$ for $K = 0$ and $V(r) = T$ for $K = 2$ the interaction can be written as

$$\begin{aligned} & A_{2\pi}(4\pi)^2 \sum_{K_2}\sum_{K_3} < 1010|K_2 0 >< 1010|K_3 0 > \frac{9}{\hat{K}_2\hat{K}_3} < \ell_1 0 \ell_2 0|K_2 0 >< \ell_3 0 \ell_4 0|K_3 0 > \\ & \frac{2}{\pi}\int q_2^2 dq_2 V^{K_2}(q_2) \hat{\ell}_1\hat{\ell}_2 (i)^{(\ell_1-\ell_2-K_2)} j_{\ell_1}(q_2 r_1) j_{\ell_2}(q_2 r_2) \\ & \frac{2}{\pi}\int q_3^2 dq_3 V^{K_3}(q_3) \hat{\ell}_4\hat{\ell}_3 (i)^{(\ell_4-\ell_3-K_3)} j_{\ell_4}(q_3 r_1) j_{\ell_3}(q_3 r_3) \\ & \Big[\left[Y_{\ell_1}(\hat{r}_1) \otimes Y_{\ell_2}(\hat{r}_2)\right]^{(K_2)} \otimes \left[\sigma_1 \otimes \sigma_2\right]^{(K_2)}\Big]^{(0)} \Big[\left[Y_{\ell_4}(\hat{r}_1) \otimes Y_{\ell_3}(\hat{r}_3)\right]^{(K_3)} \otimes \left[\sigma_1 \otimes \sigma_3\right]^{(K_3)}\Big]^{(0)} \end{aligned} \tag{2.26}$$

Here K_2 or K_3 are either 0, for the $\sigma \cdot \sigma$-term or 2, for the tensor term. These need to be recoupled:

$$\begin{aligned} & \Big[\left[Y_{\ell_1}(\hat{r}_1) \otimes Y_{\ell_2}(\hat{r}_2)\right]^{(K_2)} \otimes \left[\sigma_1 \otimes \sigma_2\right]^{(K_2)}\Big]^{(0)} \Big[\left[Y_{\ell_4}(\hat{r}_1) \otimes Y_{\ell_3}(\hat{r}_3)\right]^{(K_3)} \otimes \left[\sigma_1 \otimes \sigma_3\right]^{(K_3)}\Big]^{(0)} \\ = & (-)^{\ell_2+\lambda_2}(-)^{\ell_3+\lambda_3} \hat{K}_2\hat{\lambda}_2\hat{K}_3\hat{\lambda}_3 \begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda_2 \end{Bmatrix} \begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_3 \end{Bmatrix} \\ & \Big[\left[Y_{\ell_1}(\hat{r}_1) \otimes \sigma_1\right]^{(\lambda_2)} \otimes \left[Y_{\ell_2}(\hat{r}_2) \otimes \sigma_2\right]^{(\lambda_2)}\Big]^{(0)} \Big[\left[Y_{\ell_4}(\hat{r}_1) \otimes \sigma_1\right]^{(\lambda_3)} \otimes \left[Y_{\ell_3}(\hat{r}_3) \otimes \sigma_3\right]^{(\lambda_3)}\Big]^{(0)} \end{aligned} \tag{2.27}$$

This, in turn we can write as

$$
\begin{aligned}
&= (-)^{\ell_2+\lambda_2}(-)^{\ell_3+\lambda_3}\hat{K}_2\hat{\lambda}_2\hat{K}_3\hat{\lambda}_3\frac{\hat{J}}{\hat{\lambda}_2\hat{\lambda}_3}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}\\
&\qquad\left[\left[\left[Y_{\ell_1}(\hat{r}_1)\otimes\sigma_1\right]^{(\lambda_2)}\otimes\left[Y_{\ell_4}(\hat{r}_1)\otimes\sigma_1\right]^{(\lambda_3)}\right]^{(J)}\otimes\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda_2)}\otimes\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda_3)}\right]^{(J)}\right]^{(0)}\\
&= (-)^{\ell_2+\lambda_2+\ell_3+\lambda_3}\hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{J}\hat{L}\hat{S}\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda_3\\ L & S & J\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}\\
&\qquad\left[\left[\left[Y_{\ell_1}(\hat{r}_1)\otimes Y_{\ell_4}(\hat{r}_1)\right]^{L}\otimes\left[\sigma_1\otimes\sigma_1\right]^{(S)}\right]^{(J)}\otimes\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda_2)}\otimes\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda_3)}\right]^{(J)}\right]^{(0)}
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
&= (-)^{\ell_2+\lambda_2+\ell_3+\lambda_3}\hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{J}\hat{\ell}_1\hat{\ell}_4\hat{S}\frac{1}{\sqrt{4\pi}}\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda_3\\ L & S & J\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}\\
&\qquad\left[\left[Y_L(\hat{r}_1)\otimes\left[\sigma_1\otimes\sigma_1\right]^{(S)}\right]^{(J)}\otimes\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda_2)}\otimes\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda_3)}\right]^{(J)}\right]^{(0)}
\end{aligned}
\tag{2.29}
$$

As the reduced one-body matrix element of $(Y_\ell\otimes\sigma)^\lambda$ vanishes for $\lambda=0$, only the first matrix element in (2.2) gives a contribution. For it we have $S=0$ and $L=0$. This implies $J=0$ and $\lambda_2=\lambda_3$ as well as $\ell_1=\ell_4$. This term leads to a density dependent finite range Migdal g or tensor interaction. For this case we evaluate the 9-j symbol and simplify the interaction to

$$
\frac{1}{4\pi}(-)^{(\ell_2+\ell_3+\lambda+1)}\hat{K}_2\hat{K}_3\hat{\ell}_1\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_3 & K_3\\ 1 & 1 & \lambda\end{Bmatrix}\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda)}\odot\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda)}\right]^{(0)}
\tag{2.30}
$$

By using Eq. (2.1) of Reference [3] we write the matrix element as

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff}|(b_2\bar{b}_1)_\lambda >= 9A_{2\pi}\\
&\qquad\sum_{K_2=0,2}\sum_{K_3=0,2}\hat{\ell}_1\hat{\ell}_2\hat{\ell}_1\hat{\ell}_3 < 1010|K_20 >< 1010|K_30 >< \ell_10\ell_20|K_20 >< \ell_10\ell_30|K_30 >\\
&\qquad(i)^{(\ell_2-K_2+\ell_3-K_3)}(-)^{(\ell_1+\lambda+1)}\hat{\ell}_1\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_3 & K_3\\ 1 & 1 & \lambda\end{Bmatrix}\\
&\qquad\int r_1^2dr_1R_h(r_1)R_p(r_1)\frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_2)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} >\\
&\qquad\frac{2}{\pi}\int q_2^2dq_2V^{K_2}(q_2)\int r_2^2dr_2j_{\ell_1}(q_2r_1)j_{\ell_2}(q_2r_2)R_{a_1}(r_2)R_{a_2}(r_2)\\
&\qquad\frac{2}{\pi}\int q_3^2dq_3V^{K_3}(q_3)\int r_3^2dr_3j_{\ell_1}(q_3r_1)j_{\ell_3}(q_3r_3)R_{b_1}(r_3)R_{b_2}(r_3)
\end{aligned}
\tag{2.31}
$$

Exchanging variables 2 and 3 results in the identical expression. Thus, the commutator vanishes for this density dependent term, whereas the anti-commutator obtains a factor of 2. Further, we use $\{\vec{\tau}_1\vec{\tau}_2,\vec{\tau}_1\vec{\tau}_3\}=$

$2\vec{\tau}_2\vec{\tau}_3$. Again, we write the integrals as sums over Gauss-points

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff}|(b_2\bar{b}_1)_\lambda >= 4\cdot 9A_{2\pi} \ \ < \vec{\tau}_2\vec{\tau}_3 > \\
&\quad \sum_{K_2=0,2}\sum_{K_3=0,2} \hat{\ell}_1\hat{\ell}_2\hat{\ell}_1\hat{\ell}_3 < 1010|K_20 >< 1010|K_30 >< \ell_10\ell_20|K_20 >< \ell_10\ell_30|K_30 > \\
&\quad (i)^{(\ell_2-K_2+\ell_3-K_3)}(-)^{(\lambda+1)}\begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_3 & K_3 \\ 1 & 1 & \lambda \end{Bmatrix} \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_2)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} > \\
&\quad D^{\ell_1}_{m,s}V^{K_2,\ell_1\ell_2}_{m,n}V^{K_3,\ell_1\ell_3}_{s,t}\bar{H}^{\ell_2}_n(r_i)r_i^2w_i\bar{H}^{\ell_3}_t(r_j)r_j^2w_j \quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.32}
$$

where we have used the same definition of D as before (3.17) and

$$
V^{K,\ell_1\ell_2}_{m,n} = \sum q_k^2w_k\bar{H}^{\ell_1}_m(q_k)\bar{H}^{\ell_2}_n(q_k)V^K(q_k)
\tag{2.33}
$$

The selection rules require $\ell_1+\ell_2 = even$ and $\ell_1+\ell_3 = even$ which in turn requires $\ell_2+\ell_3 = even$. Further, ℓ_1,ℓ_2,ℓ_3 are all restricted to values of $\lambda, \lambda\pm 1$.

We now discuss the four cases separately. The first case, $K_2 = 0$,$K_3 = 0$ leads to a density dependent $\sigma_1\sigma_2$ interaction:

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,\sigma}|(b_2\bar{b}_1)_\lambda >= 4A_{2\pi} \ \ < \vec{\tau}_2\vec{\tau}_3 > D^{\ell}_{m,s}V^{0,\ell\ell}_{m,n}V^{0,\ell\ell}_{s,t}\bar{H}^{\ell}_n(r_i)r_i^2w_i\bar{H}^{\ell}_t(r_j)r_j^2w_j \\
&\quad (-)^{(\ell+\lambda+1)}\frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell)}\sigma]^\lambda\|j_{b_1} > \\
&\quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.34}
$$

second case ($K_2 = 0$,$K_3 = 2$):

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,tensor}|(b_2\bar{b}_1)_\lambda >= 4A_{2\pi} \ \ < \vec{\tau}_2\vec{\tau}_3 > \ \ \sqrt{6}\hat{\ell}_1\hat{\ell}_3(i)^{(\ell_3+\ell_1)} < \ell_10\ell_30|20 > \begin{Bmatrix} \ell_1 & \ell_3 & 2 \\ 1 & 1 & \lambda \end{Bmatrix} \\
&\quad D^{\ell_1}_{m,s}V^{0,\ell_1\ell_1}_{m,n}V^{2,\ell_1\ell_3}_{s,t}\bar{H}^{\ell_1}_n(r_i)r_i^2w_i\bar{H}^{\ell_3}_t(r_j)r_j^2w_j \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_1)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} > \\
&\quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.35}
$$

third case ($K_2 = 2$,$K_3 = 0$):

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,tensor}|(b_2\bar{b}_1)_\lambda >= 4A_{2\pi} \ \ < \vec{\tau}_2\vec{\tau}_3 > \ \ \sqrt{6}\hat{\ell}_1\hat{\ell}_3 < \ell_10\ell_30|20 > (i)^{(\ell_3+\ell_1)}\begin{Bmatrix} \ell_1 & \ell_3 & 2 \\ 1 & 1 & \lambda \end{Bmatrix} \\
&\quad D^{\ell_3}_{m,s}V^{2,\ell_3\ell_1}_{m,n}V^{0,\ell_3\ell_3}_{s,t}\bar{H}^{\ell_1}_n(r_i)r_i^2w_i\bar{H}^{\ell_3}_t(r_j)r_j^2w_j \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_1)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} > \\
&\quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{a_1}(r_j)\bar{R}_{a_2}(r_j)
\end{aligned}
\tag{2.36}
$$

This term is similar to the previous term. They represent a density dependent tensor interaction. The last term, the tensor squared term has ($K_2 = 2$,$K_3 = 2$):

$$
\begin{aligned}
&24A_{2\pi} \ \vec{\tau}_2\vec{\tau}_3 \ \ \hat{\ell}_1\hat{\ell}_2\hat{\ell}_1\hat{\ell}_3 < \ell_10\ell_20|20 >< \ell_10\ell_30|20 > \\
&\quad (i)^{(\ell_2+\ell_3)}(-)^{(\lambda+1)}\begin{Bmatrix} \ell_1 & \ell_2 & 2 \\ 1 & 1 & \lambda \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_3 & 2 \\ 1 & 1 & \lambda \end{Bmatrix} \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_2)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_1}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_2} > \\
&\quad D^{\ell_1}_{m,s}V^{2,\ell_1\ell_2}_{m,n}V^{2,\ell_1\ell_3}_{s,t}\bar{H}^{\ell_2}_n(r_i)r_i^2w_i\bar{H}^{\ell_3}_t(r_j)r_j^2w_j \quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.37}
$$

From this case we can split up terms that are similar to the previous ones. We write

$$
\begin{aligned}
&24A_{2\pi} \ < \vec{\tau}_2\vec{\tau}_3 > \quad \hat{\ell}_1\hat{\ell}_2\hat{\ell}_1\hat{\ell}_3 < \ell_1 0 \ell_2 0|20 >< \ell_1 0 \ell_3 0|20 > \\
&\quad (i)^{(\ell_2+\ell_3)} \sum_k (-)^{(\ell_1+\ell_2+\ell_3+k)}(2k+1) \begin{Bmatrix} \ell_2 & \ell_3 & k \\ 1 & 1 & \lambda \end{Bmatrix} \begin{Bmatrix} \ell_2 & \ell_3 & k \\ 2 & 2 & \ell_1 \end{Bmatrix} \begin{Bmatrix} 2 & 2 & k \\ 1 & 1 & 1 \end{Bmatrix} \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_2)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} > \\
&\quad D^{\ell_1}_{m,s} V^{2,\ell_1\ell_2}_{m,n} V^{2,\ell_1\ell_3}_{s,t} \bar{H}^{\ell_2}_n(r_i) r_i^2 w_i \bar{H}^{\ell_3}_t(r_j) r_j^2 w_j \quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.38}
$$

The term with $k = 0$ can be combined with the term (2.34). Also, the term with $k = 2$ can be combined with the terms (2.35) and (2.36).

Thus, we end up with three terms. The first corresponds to a density dependent $\sigma_1\sigma_2$ term and can be added directly to that term with the form:

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,\sigma}|(b_2\bar{b}_1)_\lambda >= 4A_{2\pi} \ < \vec{\tau}_2\vec{\tau}_3 > \bar{H}^{\ell}_n(r_i) r_i^2 w_i \bar{H}^{\ell}_t(r_j) r_j^2 w_j \\
&\quad \left[D^{\ell}_{m,s} V^{0,\ell\ell}_{m,n} V^{0,\ell\ell}_{s,t} + \sum_{\ell_1} \frac{2}{5}(2\ell_1+1) < \ell_1 0 \ell 0|20 >^2 D^{\ell_1,a}_{m,s} V^{2,\ell_1\ell}_{s,t} V^{2,\ell_1\ell}_{m,n} \right] \\
&\quad (-)^{(\ell+\lambda+1)} \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell)}\sigma]^\lambda\|j_{b_1} > \\
&\quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.40}
$$

The second corresponds to a density dependent tensor interaction:

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,tensor}|(b_2\bar{b}_1)_\lambda >= 4A_{2\pi} \ < \vec{\tau}_2\vec{\tau}_3 > \quad \bar{H}^{\ell_1}_n(r_i) r_i^2 w_i \bar{H}^{\ell_3}_t(r_j) r_j^2 w_j \\
&\quad \Bigg[D^{\ell_3}_{m,s} V^{2,\ell_3\ell_1}_{m,n} V^{0,\ell_3\ell_3}_{s,t} + D^{\ell_1}_{m,s} V^{0,\ell_1\ell_1}_{m,n} V^{2,\ell_1\ell_3}_{s,t} + 5\sqrt{6} \sum_{\ell} (2\ell+1)(-)^\ell \\
&\quad \frac{< \ell 0 \ell_1 0|20 >< \ell 0 \ell_3 0|20 >}{< \ell_1 0 \ell_3 0|20 >} \begin{Bmatrix} \ell_1 & \ell_3 & 2 \\ 2 & 2 & \ell \end{Bmatrix} \begin{Bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{Bmatrix} D^{\ell}_{m,s} V^{2,\ell\ell_1}_{m,n} V^{2,\ell\ell_3}_{s,t} \Bigg] \\
&\quad \sqrt{6}\hat{\ell}_1\hat{\ell}_3 < \ell_1 0 \ell_3 0|20 > (i)^{(\ell_3+\ell_1)} \begin{Bmatrix} \ell_1 & \ell_3 & 2 \\ 1 & 1 & \lambda \end{Bmatrix} \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell_1)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell_3)}\sigma]^\lambda\|j_{b_1} > \\
&\quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.41}
$$

The remaining term is:

$$
\begin{aligned}
&< (a_1\bar{a}_2)_\lambda|V^{eff,k=1}|(b_2\bar{b}_1)_\lambda >= 72A_{2\pi} \ < \vec{\tau}_2\vec{\tau}_3 > \\
&\quad \hat{\ell}_1\hat{\ell}\hat{\ell}_1\hat{\ell} < \ell_1 0 \ell 0|20 >^2 (-)^{(\ell+\ell_1+1)} \begin{Bmatrix} \ell & \ell & 1 \\ 1 & 1 & \lambda \end{Bmatrix} \begin{Bmatrix} \ell & \ell & 1 \\ 2 & 2 & \ell_1 \end{Bmatrix} \begin{Bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \\
&\quad \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{a_1}} < j_{a_1}\|[Y^{(\ell)}\sigma]^\lambda\|j_{a_2} > \frac{\sqrt{4\pi}}{\hat{\lambda}}(-)^{k_{b_2}} < j_{b_2}\|[Y^{(\ell)}\sigma]^\lambda\|j_{b_1} > \\
&\quad D^{\ell_1}_{m,s} V^{2,\ell_1\ell}_{m,n} V^{2,\ell_1\ell}_{s,t} \bar{H}^{\ell}_n(r_i) r_i^2 w_i \bar{H}^{\ell}_t(r_j) r_j^2 w_j \quad \bar{R}_{a_1}(r_i)\bar{R}_{a_2}(r_i)\bar{R}_{b_1}(r_j)\bar{R}_{b_2}(r_j)
\end{aligned}
\tag{2.41}
$$

3. Exchange matrix elements.

In this section we work out the exchange matrix element which we define as

$$V^{tni,x}_{p_1h_1;h_2p_2} = -V^{tni,a}_{h,p_1,p_2;h_1,p,h_2} \tag{3.1}$$

We assume the interaction can be written as sum over terms each having the form:

$$-\Big[T_1^{(\lambda_1)}(1)\otimes\big[T_2^{(\lambda_2)}\otimes T_3^{(\lambda_3)}\big]^{(\lambda_1)}\Big]^{(0)} \tag{3.2}$$

This leads to the exchange matrix element as

$$\begin{aligned}V^{tni,x}_{p_1h_1;h_2p_2} = -\big\{ &< h(1)p_1(2)p_2(3)|\Big[T_1^{(\lambda_1)}(1)\otimes\big[T_2^{(\lambda_2)}\otimes T_3^{(\lambda_3)}\big]^{(\lambda_1)}\Big]^{(0)}|h_1(1)p(2)h_2(3)>\\ &+ < h(2)p_1(3)p_2(1)|\Big[T_1^{(\lambda_1)}(1)\otimes\big[T_2^{(\lambda_2)}\otimes T_3^{(\lambda_3)}\big]^{(\lambda_1)}\Big]^{(0)}|h_1(2)p(3)h_2(1)>\\ &+ < h(3)p_1(1)p_2(2)|\Big[T_1^{(\lambda_1)}(1)\otimes\big[T_2^{(\lambda_2)}\otimes T_3^{(\lambda_3)}\big]^{(\lambda_1)}\Big]^{(0)}|h_1(3)p(1)h_2(2)>\big\}\end{aligned} \tag{3.3}$$

Here the three terms arise from the cyclic permutations. Using our phase convention for *ph*-*ph* angular momentum coupling given by Eqs. (1.4, 1.14) of Reference [3], we can carry out the summation over all m's except $m_p = m_h$, using the Wigner-Eckart theorem we obtain:

$$\begin{aligned}V^{tni,x,\lambda}_{p_1h_1,h_2p_2} = (-)^{(k_{p_2}+k_{h_1}+\lambda_1+\lambda_2+\lambda)}\frac{\hat\lambda_1\hat\lambda_2}{\hat\lambda}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ j_{p_1} & j_{h_1} & j_h\end{Bmatrix}\times\Big\{&\frac{1}{\hat\lambda_1}\langle h\|T_1^{\lambda_1}\|h_1\rangle\frac{1}{\hat\lambda_2}\langle p_1\|T_2^{\lambda_2}\|h\rangle\frac{1}{\hat\lambda}\langle p_2\|T_3^{\lambda}\|h_2\rangle\\ &+\frac{1}{\hat\lambda}\langle p_2\|T_2^{\lambda}\|h_2\rangle\frac{1}{\hat\lambda_1}\langle h\|T_3^{\lambda_1}\|h_1\rangle\frac{1}{\hat\lambda_2}\langle p_1\|T_1^{\lambda_2}\|h\rangle\\ &+\frac{1}{\hat\lambda_2}\langle p_1\|T_3^{\lambda_2}\|h\rangle\frac{1}{\hat\lambda}\langle p_2\|T_1^{\lambda}\|h_2\rangle\frac{1}{\hat\lambda_1}\langle h\|T_2^{\lambda_1}\|h_1\rangle\Big\}\end{aligned} \tag{3.4}$$

3a. Short range repulsion term.

We first turn to the correction term due to the short range repulsion as the most simple contribution of V^{tni}. From section 2 we take the interaction as

$$\begin{aligned}T^2(r_{12})T^2(r_{13}) =&(4\pi)^2\frac{1}{\sqrt{4\pi}}\sum_{\ell_1,\ell_2,\ell 3}(-)^{(\ell_2+\ell_3)}\hat\ell_2\hat\ell_3 < \ell_20\ell_30|\ell_10>\\ &\times\frac{2}{\pi}\int q_2^2dq_2\tilde T(q_2)j_{\ell_2}(q_2r_1)j_{\ell_2}(q_2r_2)\times\frac{2}{\pi}\int q_3^2dq_3\tilde T(q_3)j_{\ell_3}(q_3r_1)j_{\ell_3}(q_3r_3)\\ &\times\Big[Y_{\ell_1}(\hat r_1)\otimes\big[Y_{\ell_2}(\hat r_2)\otimes Y_{\ell_3}(\hat r_3)\big]^{(\ell_1)}\Big]^{(0)}\end{aligned} \tag{3.5}$$

Combining this with Eq. (3.4) and considering the λ's were renamed in order to obtain (3.4) we find:

$$\begin{aligned}V^{tni,x,\lambda}_{p_1h_1,h_2p_2} =&(-)^{(k_{p_1}+k_h+k_{h_1}+\lambda)}\frac{(2\ell_1+1)(2\ell_2+1)}{\hat\lambda}\begin{Bmatrix}\ell_1 & j_{h_1} & j_h\\ j_{p_1} & \ell_2 & \lambda\end{Bmatrix} < \ell_10\ell_20|\lambda0>\\ &\times\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat\lambda} < p_2\|Y_\lambda\|h_2>\Big]\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat\ell_1} < h\|Y_{\ell_1}\|h_1>\Big]\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat\ell_2} < p_1\|Y_{\ell_2}\|p>\Big]\\ &\times\Big\{\int r_1^2dr_1R_h(r_1)R_{h_1}(r_1)\\ &\qquad\frac{2}{\pi}\int q_2^2dq_2\tilde T(q_2)j_{\ell_2}(q_2r_1)\int r_2^2dr_2R_{p_1}(r_2)R_p(r_2)j_{\ell_2}(q_2r_2)\\ &\qquad\frac{2}{\pi}\int q_3^2dq_3\tilde T(q_3)j_\lambda(q_3r_1)\int r_3^2d3_2R_{p_2}(r_3)R_{h_2}(r_3)j_\lambda(q_3r_3)\end{aligned}$$

$$
\begin{aligned}
&+\int r_1^2 dr_1 R_{p_2}(r_1) R_{h_2}(r_1) \\
&\qquad \frac{2}{\pi}\int q_2^2 dq_2 \tilde{T}(q_2) j_{\ell_1}(q_2 r_1)\int r_2^2 dr_2 R_{h_1}(r_2) R_h(r_2) j_{\ell_1}(q_2 r_2) \\
&\qquad \frac{2}{\pi}\int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell_2}(q_3 r_1)\int r_3^2 d3_2 R_{p_1}(r_3) R_p(r_3) j_{\ell_2}(q_3 r_3) \\
&+\int r_1^2 dr_1 R_{p_1}(r_1) R_p(r_1) \\
&\qquad \frac{2}{\pi}\int q_2^2 dq_2 \tilde{T}(q_2) j_\lambda(q_2 r_1)\int r_2^2 dr_2 R_{p_2}(r_2) R_{h_2}(r_2) j_\lambda(q_2 r_2) \\
&\qquad \frac{2}{\pi}\int q_3^2 dq_3 \tilde{T}(q_3) j_{\ell_1}(q_3 r_1)\int r_3^2 d3_2 R_{h_1}(r_3) R_h(r_3) j_{\ell_1}(q_3 r_3)\Big\}
\end{aligned}
\tag{3.6}
$$

Using Gauss integration, we write the radial matrix element as

$$
\begin{aligned}
\Big\{ & r_k^2 w_k R_h(r_k) R_{h_1}(r_k) \bar{H}_m^{\ell_2}(r_k) T_{mn}^{\ell_2} \bar{R}_{p_1}(r_i) \bar{R}_p(r_i) \bar{H}_n^{\ell_2}(r_i) r_i^2 w_i \\
& \qquad \bar{H}_s^\lambda(r_k) T_{st}^\lambda \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \bar{H}_t^\lambda(r_j) r_j^2 w_j \\
+ & r_k^2 w_k R_p(r_k) R_{p_1}(r_k) \bar{H}_m^{\ell_1}(r_k) T_{mn}^{\ell_1} \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) \bar{H}_n^{\ell_1}(r_i) r_i^2 w_i \\
& \qquad \bar{H}_s^\lambda(r_k) T_{st}^\lambda \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \bar{H}_t^\lambda(r_j) r_j^2 w_j \\
+ & r_k^2 w_k R_{p_2}(r_k) R_{h_2}(r_k) \bar{H}_m^{\ell_1}(r_k) T_{mn}^{\ell_1} \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) \bar{H}_n^{\ell_1}(r_i) r_i^2 w_i \\
& \qquad \bar{H}_s^{\ell_2}(r_k) T_{st}^{\ell_2} \bar{R}_{p_1}(r_j) \bar{R}_p(r_j) \bar{H}_t^{\ell_2}(r_j) r_j^2 w_j \Big\}
\end{aligned}
\tag{3.7}
$$

defining the kernel

$$
G^\ell(r_k, r_i) = \sqrt{r_k^2 w_k}\, \bar{H}_m^\ell(r_k) T_{mn}^\ell \bar{H}_n^\ell(r_i) r_i^2 w_i \tag{3.8}
$$

allows us to write the radial integrals as

$$
\begin{aligned}
\Big\{ & R_h(r_k) R_{h_1}(r_k) \bar{R}_{p_1}(r_i) \bar{R}_p(r_i) G^{\ell_2}(r_k, r_i) G^\lambda(r_k, r_j) \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \\
+ & R_p(r_k) R_{p_1}(r_k) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) G^{\ell_1}(r_k, r_i) G^\lambda(r_k, r_j) \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \\
+ & R_{p_2}(r_k) R_{h_2}(r_k) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) G^{\ell_1}(r_k, r_i) G^{\ell_2}(r_k, r_j) \bar{R}_{p_1}(r_j) \bar{R}_p(r_j) \Big\}
\end{aligned}
\tag{3.9}
$$

3b. Two-pion exchange term.

Again, we take the interaction from section 2 as

$$
\begin{aligned}
&A_{2\pi}(4\pi)^{3/2}\sum_{K_2=0,2}\sum_{K_3=0,2} <1010|K_20><1010|K_30>(-)^{\ell_2+\lambda_2+\ell_3+\lambda_3}\\
&\qquad \frac{2}{\pi}\int q_2^2dq_2\tilde{V}^{K_2}(q_2)\hat{\ell}_1\hat{\ell}_2(i)^{(\ell_1-\ell_2-K_2)}j_{\ell_1}(q_2r_1)j_{\ell_2}(q_2r_2)\frac{3}{\hat{K}_2}<\ell_10\ell_20|K_20>\\
&\qquad \frac{2}{\pi}\int q_3^2dq_3\tilde{V}^{K_3}(q_3)\hat{\ell}_4\hat{\ell}_3(i)^{(\ell_4-\ell_3-K_3)}j_{\ell_4}(q_3r_1)j_{\ell_3}(q_3r_3)\frac{3}{\hat{K}_3}<\ell_40\ell_30|K_30>\\
&\qquad \hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_1\hat{\ell}_1\hat{\ell}_4\hat{S}\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda_3\\ L & S & \lambda_1\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}\langle\ell_10\ell_40|L0\rangle\\
&\qquad \left[\left[Y_L(\hat{r}_1)\otimes\left[\sigma_1\otimes\sigma_1\right]^{(S)}\right]^{(\lambda_1)}\otimes\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda_2)}\otimes\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda_3)}\right]^{(\lambda_1)}\right]^{(0)}
\end{aligned}
\tag{3.8}
$$

Here K_2 or K_3 are either 0, for the $\sigma{\cdot}\sigma$-term or 2, for the tensor term. Further, S is 0 in the anti-commutator term and 1 in the commutator term. As for the present correction we only have a contribution from the anticommutator, we write the interaction as twice the S=0 contribution:

$$
\begin{aligned}
&18A_{2\pi}(4\pi)^{3/2}\sum_{K_2=0,2}\sum_{K_3=0,2} <1010|K_20><1010|K_30>(-)^{\lambda_2+\ell_3+\lambda_1}\\
&\quad \frac{2}{\pi}\int q_2^2dq_2\tilde{V}^{K_2}(q_2)\hat{\ell}_1\hat{\ell}_2(i)^{(\ell_1-\ell_2-K_2)}j_{\ell_1}(q_2r_1)j_{\ell_2}(q_2r_2)<\ell_10\ell_20|K_20>\\
&\quad \frac{2}{\pi}\int q_3^2dq_3\tilde{V}^{K_3}(q_3)\hat{\ell}_4\hat{\ell}_3(i)^{(\ell_4-\ell_3-K_3)}j_{\ell_4}(q_3r_1)j_{\ell_3}(q_3r_3)<\ell_40\ell_30|K_30>\\
&\quad \hat{\lambda}_2\hat{\lambda}_3\hat{\ell}_1\hat{\ell}_4\begin{Bmatrix}\lambda_2 & \ell_1 & 1\\ \ell_4 & \lambda_3 & \lambda_1\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}\langle\ell_10\ell_40|\lambda_10\rangle\\
&\quad \left[Y_{\lambda_1}(\hat{r}_1)\otimes\left[\left[Y_{\ell_2}(\hat{r}_2)\otimes\sigma_2\right]^{(\lambda_2)}\otimes\left[Y_{\ell_3}(\hat{r}_3)\otimes\sigma_3\right]^{(\lambda_3)}\right]^{(\lambda_1)}\right]^{(0)}
\end{aligned}
\tag{3.9}
$$

We carry out the summation over all m's and do the summation over cyclic permutations by using Eq. (3.4). However, it should be noted that in the second and third term of the cyclic permutations the λ's in Eq. (3.4) were renamed

$$
\begin{aligned}
&18A_{2\pi}<1010|K_20><1010|K_30>\hat{\ell}_1\hat{\ell}_4\hat{\ell}_1\hat{\ell}_2\hat{\ell}_4\hat{\ell}_3\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ j_{p_1} & j_{h_1} & j_h\end{Bmatrix}(-)^{\ell_3}\\
&\quad (i)^{(\ell_1-\ell_2-K_2+\ell_4-\ell_3-K_3)}<\ell_10\ell_20|K_20><\ell_40\ell_30|K_30>(-)^{(k_h+k_{p_1}+k_{h_1})}\\
&\quad \frac{2}{\pi}\int q_2^2dq_2\tilde{V}^{K_2}(q_2)j_{\ell_1}(q_2r_1)j_{\ell_2}(q_2r_2)\\
&\quad \frac{2}{\pi}\int q_3^2dq_3\tilde{V}^{K_3}(q_3)j_{\ell_4}(q_3r_1)j_{\ell_3}(q_3r_3)\\
&\Big\{\hat{\lambda}_2\hat{\lambda}\begin{Bmatrix}\lambda_2 & \ell_1 & 1\\ \ell_4 & \lambda & \lambda_1\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda\end{Bmatrix}\langle\ell_10\ell_40|\lambda_10\rangle(-)^{\lambda}\\
&\quad \left[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|Y^{\lambda_1}\|h_1\rangle\right]\left[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|[Y_{\ell_2}\sigma]^{\lambda_2}\|p\rangle\right]\left[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|[Y_{\ell_3}\sigma]^{\lambda}\|h_2\rangle\right]\\
&+\hat{\lambda}_1\hat{\lambda}_2\begin{Bmatrix}\lambda_1 & \ell_1 & 1\\ \ell_4 & \lambda_2 & \lambda\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_1\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_2\end{Bmatrix}\langle\ell_10\ell_40|\lambda0\rangle(-)^{\lambda_2}\\
&\quad \left[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|Y^{\lambda}\|h_2\rangle\right]\left[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|[Y_{\ell_2}\sigma]^{\lambda_1}\|h_1\rangle\right]\left[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|[Y_{\ell_3}\sigma]^{\lambda_2}\|p\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{\lambda}\hat{\lambda}_1\begin{Bmatrix} \lambda & \ell_1 & 1 \\ \ell_4 & \lambda_1 & \lambda_2 \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda \end{Bmatrix}\begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_1 \end{Bmatrix}\langle \ell_1 0 \ell_4 0|\lambda_2 0\rangle(-)^{\lambda_1} \\
&\quad\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|Y^{\lambda_2}\|p\rangle\Big]\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|[Y_{\ell_2}\sigma]^{\lambda}\|h_2\rangle\Big]\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|[Y_{\ell_3}\sigma]^{\lambda_1}\|h_1\rangle\Big]\Bigg\}
\end{aligned} \tag{3.10}
$$

We write the radial integrals as summations over the Gauss points. We compute the matrix element as

$$
\begin{aligned}
V^{tni,x,\lambda}_{p_1h_1,h_2p_2} =& 18A_{2\pi} < 1010|K_2 0 >< 1010|K_3 0 > \hat{\ell}_1\hat{\ell}_4\hat{\ell}_1\hat{\ell}_2\hat{\ell}_4\hat{\ell}_3\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix} \\
&(i)^{(\ell_1-\ell_2-K_2+\ell_4-\ell_3-K_3)} < \ell_1 0\ell_2 0|K_2 0 >< \ell_4 0\ell_3 0|K_3 0 > (-)^{(k_h+k_{p_1}+k_{h_1})} \\
&\Bigg\{\hat{\lambda}_2\hat{\lambda}\begin{Bmatrix} \lambda_2 & \ell_1 & 1 \\ \ell_4 & \lambda & \lambda_1 \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda_2 \end{Bmatrix}\begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda \end{Bmatrix}\langle \ell_1 0 \ell_4 0|\lambda_1 0\rangle(-)^{\lambda+\ell_3} \\
&\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|Y^{\lambda_1}\|h_1\rangle\Big]\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|[Y_{\ell_2}\sigma]^{\lambda_2}\|p\rangle\Big]\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|[Y_{\ell_3}\sigma]^{\lambda}\|h_2\rangle\Big] \\
&r_k^2 w_k R_h(r_k)R_{h_1}(r_k)\bar{H}^{\ell_1}_m(r_k)T^{K_2,\ell_1,\ell_2}_{mn}\bar{R}_{p_1}(r_i)\bar{R}_p(r_i)\bar{H}^{\ell_2}_n(r_i)r_i^2 w_i \\
&\quad\bar{H}^{\ell_4}_s(r_k)T^{K_3,\ell_4,\ell_3}_{st}\bar{R}_{p_2}(r_j)\bar{R}_{h_2}(r_j)\bar{H}^{\ell_3}_t(r_j)r_j^2 w_j \\
&+\hat{\lambda}_1\hat{\lambda}_2\begin{Bmatrix} \lambda_1 & \ell_1 & 1 \\ \ell_4 & \lambda_2 & \lambda \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda_1 \end{Bmatrix}\begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_2 \end{Bmatrix}\langle \ell_1 0 \ell_4 0|\lambda 0\rangle(-)^{\lambda_2+\ell_3} \\
&\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|Y^{\lambda}\|h_2\rangle\Big]\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|[Y_{\ell_2}\sigma]^{\lambda_1}\|h_1\rangle\Big]\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|[Y_{\ell_3}\sigma]^{\lambda_2}\|p\rangle\Big] \\
&r_k^2 w_k R_{p_2}(r_k)R_{h_2}(r_k)\bar{H}^{\ell_1}_m(r_k)T^{K_2,\ell_1,\ell_2}_{mn}\bar{R}_{h_1}(r_i)\bar{R}_h(r_i)\bar{H}^{\ell_2}_n(r_i)r_i^2 w_i \\
&\quad\bar{H}^{\ell_4}_s(r_k)T^{K_3,\ell_4\ell_3}_{st}\bar{R}_{p_1}(r_j)\bar{R}_p(r_j)\bar{H}^{\ell_3}_t(r_j)r_j^2 w_j \\
&+\hat{\lambda}\hat{\lambda}_1\begin{Bmatrix} \lambda & \ell_1 & 1 \\ \ell_4 & \lambda_1 & \lambda_2 \end{Bmatrix}\begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_1 \end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda \end{Bmatrix}\langle \ell_1 0 \ell_4 0|\lambda_2 0\rangle(-)^{\lambda_1+\ell_3} \\
&\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|Y^{\lambda_2}\|p\rangle\Big]\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|[Y_{\ell_2}\sigma]^{\lambda}\|h_2\rangle\Big]\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|[Y_{\ell_3}\sigma]^{\lambda_1}\|h_1\rangle\Big] \\
&r_k^2 w_k R_{p_1}(r_k)R_p(r_k)\bar{H}^{\ell_4}_m(r_k)T^{K_3,\ell_4,\ell_3}_{mn}\bar{R}_{h_1}(r_i)\bar{R}_h(r_i)\bar{H}^{\ell_3}_n(r_i)r_i^2 w_i \\
&\quad\bar{H}^{\ell_1}_s(r_k)T^{K_2,\ell_1,\ell_2}_{st}\bar{R}_{p_2}(r_j)\bar{R}_{h_2}(r_j)\bar{H}^{\ell_2}_t(r_j)r_j^2 w_j\Bigg\}
\end{aligned} \tag{3.11}
$$

We define the interaction kernel as:

$$
\begin{aligned}
K^{\lambda,\ell_1\ell_2}(r_k,r_i) :=& \sum_K (i)^{(\ell_1-\ell_2-K)} < 1010|K0 > \hat{\ell}_1\hat{\ell}_1\hat{\ell}_2 < \ell_1 0\ell_2 0|K0 > \begin{Bmatrix} \ell_1 & \ell_2 & K \\ 1 & 1 & \lambda \end{Bmatrix}\hat{\lambda} \\
&\sum_{m,n}\sqrt{r_k^2 w_k}\bar{H}^{\ell_1}_m(r_k)T^{K,\ell_1,\ell_2}_{mn}H^{\ell_2}_n(r_i)r_i^2 w_i
\end{aligned} \tag{3.12}
$$

With this definition we write the matrix element as

$$
\begin{aligned}
V^{tni,x,\lambda}_{p_1h_1,h_2p_2} =& 18A_{2\pi}\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}(-)^{(k_h+k_{p_1}+k_{h_1})}\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix} \\
&\Bigg\{\begin{Bmatrix} \lambda_2 & \ell_1 & 1 \\ \ell_4 & \lambda & \lambda_1 \end{Bmatrix}\langle \ell_1 0 \ell_4 0|\lambda_1 0\rangle(-)^{\lambda+\ell_3} \\
&\quad\Big[(-)^{k_h}\frac{\sqrt{4\pi}}{\hat{\lambda}_1}\langle h\|Y^{\lambda_1}\|h_1\rangle\Big]\Big[(-)^{k_{p_1}}\frac{\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1\|[Y_{\ell_2}\sigma]^{\lambda_2}\|p\rangle\Big]\Big[(-)^{k_{p_2}}\frac{\sqrt{4\pi}}{\hat{\lambda}}\langle p_2\|[Y_{\ell_3}\sigma]^{\lambda}\|h_2\rangle\Big] \\
&\quad R_h(r_k)R_{h_1}(r_k)K^{\lambda_2,\ell_1,\ell_2}(r_k,r_i)\bar{R}_{p_1}(r_i)\bar{R}_p(r_i)K^{\lambda,\ell_4,\ell_3}(r_k,r_j)\bar{R}_{p_2}(r_j)\bar{R}_{h_2}(r_j)
\end{aligned}
$$

$$
\begin{aligned}
&+\begin{Bmatrix} \lambda_1 & \ell_1 & 1 \\ \ell_4 & \lambda_2 & \lambda \end{Bmatrix} \langle \ell_1 0 \ell_4 0 | \lambda 0 \rangle (-)^{\lambda_2+\ell_3} \\
&\left[(-)^{k_{p_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 \| Y^{\lambda} \| h_2 \rangle\right] \left[(-)^{k_h} \frac{\sqrt{4\pi}}{\hat{\lambda}_1} \langle h \| [Y_{\ell_2}\sigma]^{\lambda_1} \| h_1 \rangle\right] \left[(-)^{k_{p_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 \| [Y_{\ell_3}\sigma]^{\lambda_2} \| p \rangle\right] \\
&R_{p_2}(r_k) R_{h_2}(r_k) K^{\lambda_1,\ell_1,\ell_2}(r_k,r_i) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) K^{\lambda_2,\ell_4,\ell_3}(r_k,r_j) \bar{R}_{p_1}(r_j) \bar{R}_p(r_j) \\
&+\begin{Bmatrix} \lambda & \ell_1 & 1 \\ \ell_4 & \lambda_1 & \lambda_2 \end{Bmatrix} \langle \ell_1 0 \ell_4 0 | \lambda_2 0 \rangle (-)^{\lambda_1+\ell_3} \\
&\left[(-)^{k_{p_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 \| Y^{\lambda_2} \| p \rangle\right] \left[(-)^{k_{p_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 \| [Y_{\ell_2}\sigma]^{\lambda} \| h_2 \rangle\right] \left[(-)^{k_h} \frac{\sqrt{4\pi}}{\hat{\lambda}_1} \langle h \| [Y_{\ell_3}\sigma]^{\lambda_1} \| h_1 \rangle\right] \\
&R_{p_1}(r_k) R_p(r_k) K^{\lambda_1,\ell_4\ell_3}(r_k,r_i) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) K^{\lambda,\ell_1,\ell_2}(r_k,r_j) \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \Big\}
\end{aligned}
\tag{3.13}
$$

Similarly, we get the commutator contribution as

$$
\begin{aligned}
V^{tni,cx,\lambda}_{p_1h_1,h_2p_2} =& \frac{9}{2}\sqrt{\frac{3}{2}} A_{2\pi} \frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}} (-)^{(k_h+k_{p_1}+k_{h_1})} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix} \\
&\Bigg\{ \begin{Bmatrix} \ell_1 & 1 & \lambda_2 \\ \ell_4 & 1 & \lambda \\ L & 1 & \lambda_1 \end{Bmatrix} \hat{\lambda}_1 \langle \ell_1 0 \ell_4 0 | L 0 \rangle (-)^{\lambda_1+\ell_2+\ell_3} \\
&\left[(-)^{k_h} \frac{\sqrt{4\pi}}{\hat{\lambda}_1} \langle h \| [Y_L\sigma]^{\lambda_1} \| h_1 \rangle\right] \left[(-)^{k_{p_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 \| [Y_{\ell_2}\sigma]^{\lambda_2} \| p \rangle\right] \left[(-)^{k_{p_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 \| [Y_{\ell_3}\sigma]^{\lambda} \| h_2 \rangle\right] \\
&R_h(r_k) R_{h_1}(r_k) K^{\lambda_2,\ell_1,\ell_2}(r_k,r_i) \bar{R}_{p_1}(r_i) \bar{R}_p(r_i) K^{\lambda,\ell_4,\ell_3}(r_k,r_j) \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \\
&+\begin{Bmatrix} \ell_1 & 1 & \lambda_1 \\ \ell_4 & 1 & \lambda_2 \\ L & 1 & \lambda \end{Bmatrix} \hat{\lambda} \langle \ell_1 0 \ell_4 0 | L 0 \rangle (-)^{\lambda+\ell_2+\ell_3} \\
&\left[(-)^{k_{p_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 \| [Y_L\sigma]^{\lambda} \| h_2 \rangle\right] \left[(-)^{k_h} \frac{\sqrt{4\pi}}{\hat{\lambda}_1} \langle h \| [Y_{\ell_2}\sigma]^{\lambda_1} \| h_1 \rangle\right] \left[(-)^{k_{p_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 \| [Y_{\ell_3}\sigma]^{\lambda_2} \| p \rangle\right] \\
&R_{p_2}(r_k) R_{h_2}(r_k) K^{\lambda_1,\ell_1,\ell_2}(r_k,r_i) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) K^{\lambda_2,\ell_4,\ell_3}(r_k,r_j) \bar{R}_{p_1}(r_j) \bar{R}_p(r_j) \\
&+\begin{Bmatrix} \ell_1 & 1 & \lambda \\ \ell_4 & 1 & \lambda_1 \\ L & 1 & \lambda_2 \end{Bmatrix} \hat{\lambda}_2 \langle \ell_1 0 \ell_4 0 | L 0 \rangle (-)^{\lambda_2+\ell_2+\ell_3} \\
&\left[(-)^{k_{p_1}} \frac{\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 \| [Y_L\sigma]^{\lambda_2} \| p \rangle\right] \left[(-)^{k_{p_2}} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 \| [Y_{\ell_2}\sigma]^{\lambda} \| h_2 \rangle\right] \left[(-)^{k_h} \frac{\sqrt{4\pi}}{\hat{\lambda}_1} \langle h \| [Y_{\ell_3}\sigma]^{\lambda_1} \| h_1 \rangle\right] \\
&R_{p_1}(r_k) R_p(r_k) K^{\lambda_1,\ell_4\ell_3}(r_k,r_i) \bar{R}_{h_1}(r_i) \bar{R}_h(r_i) K^{\lambda,\ell_1,\ell_2}(r_k,r_j) \bar{R}_{p_2}(r_j) \bar{R}_{h_2}(r_j) \Bigg\}
\end{aligned}
\tag{3.14}
$$

References

[1] J. H. Heisenberg and B. Mihaila, “Ground state correlations and mean-field in ^{16}O: Part II,” nucl-th/9912023 (1998).

[2] J. Carlson, V.R. Pandharipande, and R.B. Wiringa, Nucl. Phys. A **401, 59 (1983).**

[3] B. Mihaila and J. H. Heisenberg, 'Computation of two-body matrix elements from the Argonne V-18 potential', nucl-th/9802012 (1998).